Rational Function Integration

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Abstract

The derivative of any rational function is a rational function. An algorithm and decision procedure for finding the rational function anti-derivative of a rational function is presented. This algorithm is then extended to derivatives of rational functions including instances of a radical involving the integration variable.

1. Rational Function Differentiation

Let

$$f(x) = \prod_{j\neq 0} p_j(x)^j \tag{1}$$

be a rational function of x where the primitive polynomials $p_j(x)$ are square-free and mutually relatively prime.

The derivative of $f(x)$ is

$$\frac{df}{dx}(x) = \sum_j j\, p_j(x)^{j-1}\, p_j'(x) \prod_{k\neq j} p_k(x)^k \tag{2}$$

Lemma 1. *Given square-free and relatively prime primitive polynomials $p_j(x)$, $\sum_j j\, p_j'(x) \prod_{k\neq j} p_k(x)$ has no factors in common with $p_j(x)$.*

Assume that the sum has a common factor $p_h(x)$ such that:

$$p_h(x) \left| \sum_j j\, p_j'(x) \prod_{k\neq j} p_k(x) \right.$$

$p_h(x)$ divides all terms for $j \neq h$. Because it divides the whole sum, $p_h(x)$ must divide the remaining term $h\, p_h'(x) \prod_{k\neq h} p_k(x)$. From the given conditions, $p_h(x)$ does not divide $p_h'(x)$ because $p_h(x)$ is square-free; and $p_h(x)$ does not divide $p_k(x)$ for $k \neq h$ because they are relatively prime.

2. Rational Function Integration

Separating square-and-higher factors from the sum in equation (2):

$$\frac{df}{dx}(x) = \left[\prod_j p_j(x)^{j-1}\right]\left[\sum_j j\,p_j'(x)\prod_{k\neq j} p_k(x)\right] \tag{3}$$

There are no common factors between the sum and product terms of equation (3) because of the relatively prime condition of equation (1) and because of Lemma 1. Hence, this equation cannot be reduced and is canonical.

Split equation (3) into factors by the sign of the exponents, giving:

$$\frac{df}{dx}(x) = \frac{\prod_{j>2} p_j(x)^{j-1}}{\prod_{j<0} p_j(x)^{1-j}}\,\overbrace{p_2(x)\sum_j j\,p_j'(x)\prod_{k\neq j} p_k(x)}^{\mathbf{L}} \tag{4}$$

The denominator is $\prod_{j\leq 0} p_j(x)^{1-j}$. Its individual $p_j(x)$ can be separated by square-free factorization. The $p_j(x)$ for $j > 2$ can also be separated by square-free factorization of the numerator. Neither $p_2(x)$ nor $\sum_j j\,p_j'(x)\prod_{k\neq j} p_k(x)$ have square factors; so square-free factorization will not separate them. Treating $p_2(x)$ as 1 lets its factor be absorbed into $p_1(x)$. Note that $p_j(x) = 1$ for factor exponents j which don't occur in the factorization of df/dx. All the $p_j(x)$ are now known except $p_1(x)$. Once $p_1(x)$ is known, $f(x)$ can be recovered by equation (1). Let polynomial L be the result of dividing the numerator of df/dx by $\prod_{j>2} p_j(x)^{j-1}$.

$$\overbrace{\sum_j j\,p_j'(x)\prod_{k\neq j} p_k(x)}^{\mathbf{L}} = \overbrace{\sum_{j\neq 1} j\,p_j'(x)\prod_{1\neq k\neq j} p_k(x)}^{\mathbf{M}}\,p_1(x) + p_1'(x)\,\overbrace{\prod_{k\neq 1} p_k(x)}^{\mathbf{N}} \tag{5}$$

Because they don't involve $p_1(x)$, polynomials M and N in equation (5) can be computed from the square-free factorizations of the numerator and denominator. This allows $p_1(x)$ to be constructed by a process resembling long division. The trick at each step is to construct a monomial $q(x)$ such that $\mathtt{M}\,q(x) + q'(x)\,\mathtt{N}$ cancels the highest term of dividend R (which is initially L).

Let $\mathtt{deg}(p)$ be the degree of x in polynomial $p \neq 0$ and $\mathtt{deg}(0) = -1$. Let $\mathtt{coeff}(p, w)$ be the coefficient of the x^w term of polynomial p for $w \geq 0$.

Note that $\mathtt{deg}(\mathtt{M}) = \mathtt{deg}(\mathtt{N}) - 1$ because the derivative of exactly one of the $p_j(x)$ occurs instead of $p_j(x)$ in each term of M. And $\mathtt{deg}(q(x)\,\mathtt{M}) = \mathtt{deg}(q'(x)\,\mathtt{N})$ because $\mathtt{deg}(q'(x)) = \mathtt{deg}(q(x)) - 1$.

The polynomial $p_1(x)$ can be constructed by the following procedure. Let A, C, and R be rational expressions. Only the numerators of A and R contain powers of x. Starting from polynomials L, M, and N:

```
A := 0;
R := L;
Nxd := deg(N);
while ( ( g := deg(num(R)) - Nxd + 1 ) >= 0 ) do
  Rxd := deg(num(R));
  RxC := coeff(num(R),Rxd);
  C := RxC / ( coeff(M,Nxd-1) + g*coeff(N,Nxd) ) / denom(R);
  A := A + C * x^g;
  R := R - C * ( M*x^g + N*diff(x^g,x) );
  if ( deg(num(R)) > Rxd ) then fail;
  if ( 0 = R ) then return ( A );
```

At the end of this process, if $\mathtt{R} = 0$, then $p_1(x)$ is the numerator of A and $f(x) = \prod_j p_j(x)^j$ divided by the denominator of A. Otherwise the anti-derivative is not a rational function.

Just as this algorithm works with $p_2(x)$ absorbed into $p_1(x)$, it works with all of the $p_j(x)$ for $j > 1$ absorbed into $p_1(x)$. This removes the need to factor the numerator and provides the opportunity to enhance the algorithm to handle algebraic field extensions.

3. Algebraic field extension

Let y be a variable representing one of the solutions of its defining equation (reduction rule) represented by a polynomial $\mathtt{Y} = 0$. For example `Y` would be $y^3 - x$ for a cube root of x.

As discussed by Caviness and Fateman[1], multiple field extensions involving the same variable can be combined into a single field extension. For the purposes of integration, combine the field extensions involving the variable of integration x into a single variable y with its defining equation `Y`.

In order to normalize polynomials with regard to `Y`, each polynomial `P` containing y is replaced by $\mathtt{prem}(\mathtt{P},\mathtt{Y})$, the remainder of pseudo-division of `P` by `Y`, as described by Knuth Volume 2[2].

While that process normalizes polynomials, it doesn't fully normalize ratios of polynomials, for instance:

$$1/y^2 = 1/(\sqrt[3]{x})^2 = \sqrt[3]{x}/x = y/x$$

After the polynomials are normalized, if the denominator still contains the field extension y, it is possible to move y to the numerator by multiplying both numerator and denominator by the y-conjugate of the denominator, then normalizing both numerator and denominator by `Y`. The conjugate of a polynomial `P` with respect to `Y` can be computed by the following procedure where $\mathtt{deg}(q)$ is the degree of y in polynomial q and $\mathtt{pquo}(\mathtt{Y},\mathtt{P})$ and $\mathtt{prem}(\mathtt{Y},\mathtt{P})$ are the quotient and remainder of pseudo-division of `Y` by `P`:

```
conj(P):
  if ( deg(P) < deg(Y) ) then
    Q := pquo(Y,P);
    R := prem(Y,P);
  else
    Q := 1;
    R := 0;
  if ( deg(R) = 0 )
    then return ( Q );
    else return ( Q * conj(R) );
```

With a single algebraic field extension y which is a function of x, and the denominator free of y, and all the numerator factors in $p_1(x,y)$, the previous development can be reformulated:

$$f(x,y) = \prod_{j\le 1} p_j(x,y)^j \tag{6}$$

The derivative of $f(x,y)$ with respect to x is

$$\frac{df}{dx}(x,y) = \sum_{j\le 1} j\, p_j(x,y)^{j-1}\, p_j'(x,y) \prod_{k\ne j} p_k(x,y)^k \tag{7}$$

Separating into numerator and denominator:

$$\frac{df}{dx}(x,y) = \frac{\sum_j j\, p_j'(x,y) \prod_{k\ne j} p_k(x,y)}{\prod_{j\le 0} p_j(x,y)^{1-j}} \tag{8}$$

This time, `L` is the whole numerator of equation (8). Note that the denominator includes $p_0(x,y)$; $p_0(x,y)$ does not contribute to `M` because its coefficient j is 0. Separating $p_1(x,y)$ from the denominator factors:

$$\overbrace{\sum_j j\, p_j'(x,y) \prod_{k\ne j} p_k(x,y)}^{\mathbf{L}} = \overbrace{\sum_{j\le 0} j\, p_j'(x,y) \prod_{k\ne j} p_k(x,y)}^{\mathbf{M}}\, p_1(x,y) + p_1'(x,y)\, \overbrace{\prod_{k\le 0} p_k(x,y)}^{\mathbf{N}}$$

Because they don't involve $p_1(x,y)$, polynomials `M` and `N` can be computed from the square-free factorization of the denominator. The trick at each step is to construct a polynomial t such that $\mathtt{M}\,t + t'\,\mathtt{N}$ cancels the highest term of dividend `R` (initial $\mathtt{R} = \mathtt{L}$).

Let `A`, `C`, and `R` be rational expressions. Let `Q` and `T` be polynomials of x containing no algebraic extensions. Let $g = \deg(\mathtt{R}, x) - \deg(\mathtt{N}, x) + 1$.

When there is no algebraic extension, $t = x^g$. If there is an algebraic extension y, let q be the denominator of normalized dy/dx, f be the integer quotient $g/\deg(q, x)$, and set g to the remainder of $g/\deg(q, x)$. Then:

$$t = q^f \, x^g \, y^h$$

The polynomial $p_1(x, y)$ can be constructed by the following procedure. Starting from polynomials `L`, `M`, and `N`:

```
A := 0;
R := L;
Q := denom( normalize( diff(y,x) ) );
Nyd := deg(N,y);
NyC := coeff(N,y,Nyd);
Nxd := deg(NyC,x);
loop
  Ryd := deg(num(R),y);
  RyC := coeff(num(R),y,Ryd);
  Rxd := deg(RyC,x);
  h := Ryd - Nyd;
  g := ( Rxd - Nxd + 1 );
  if ( 0 = deg(Q,x) )
    T := x^g;
  else
    f := quotient(g,deg(Q,x));
    g := remainder(g,deg(Q,x));
    T := Q^f * x^g * y^h;
  dT := diff(T,x);
  B := normalize( N*dT + M*T );
  C := coeff(RyC,x,Rxd) * denom(B) / denom(R)
       / coeff(coeff(num(B),y,Ryd),x,Rxd);
  A := A + C * T;
  R := R - C * B;
  if ( 0 = R ) then return ( A );
  if ( deg(num(R),y) > Ryd ) then fail;
  if ( deg(num(R),y) = Ryd and
       deg(coeff(num(R),y,deg(num(R),y)),x) >= Rxd ) then fail;
```

The looping continues only as long as the degree of `R` decreases. If this process succeeds, then $p_1(x, y)$ is the numerator of `A` and $f(x, y) = \prod_j p_j(x, y)^j$ divided by the denominator of `A`.

References

[1] B. F. Caviness and R. J. Fateman. Simplification of radical expressions. In *Proceedings of the Third ACM Symposium on Symbolic and Algebraic Computation*, SYMSAC '76, pages 329–338, New York, NY, USA, 1976. ACM.

[2] Donald E. Knuth. *The Art of Computer Programming, Volume 2 (2nd Ed.): Seminumerical Algorithms.* Addison-Wesley Longman Publishing Co., Inc., USA, 1982.